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A Best Two-Dimensional Space of Approximating Functions

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The set $\mathscr{S}$, of all real valued functions f defined on a compact metric space (M, ρ) which satisfy $|f(x) - f(y)| \leqslant \rho(x, y)$, is of great importance in approximation theory. For instance, if $M = [0, 1]$, then it can be shown that Jackson's theorem is equivalent to the statement that for every $f \in \mathscr{S}$, there is a $p \in P_n$ (the n-th degree polynomials) such that

$$\max_{x\in M} |f(x) - p(x)| \leqslant c/n.$$

In ([2], Theorem 1, p. 26), it is proven that if G is any n-dimensional space of real functions on M, then there exists an $f \in \mathscr{S}$ such that

$$\inf_{g\in G, x\in M} \sup |f(x) - g(x)| \geqslant 1/2n.$$

However, it is fairly easy to show that if G is the span of $\{T_1(x) \cdots T_n(x)\}$, where

$$T_k(x) = \begin{cases} 1, & \text{if } x \in [k - 1/n, k/n), \\ 0, & \text{otherwise,} \end{cases} \qquad k = 1 \cdots n - 1,$$

and

$$T_n(x) = \begin{cases} 1, & \text{if } x \in [n - 1/n, 1], \\ 0, & \text{otherwise,} \end{cases}$$

then if $f \in \mathscr{S}$, there is a $g \in G$ such that

$$\sup_{x\in M} |f(x) - g(x)| \leqslant 1/2n$$

(we just approximate f in each of the intervals, by $1/2[\sup f + \inf f]$). Thus, characteristic functions form a best n-dimensional approximating space on [0, 1].

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In this paper, we extend this result for $n = 2$ to arbitrary compact metric spaces. In the last section of the paper, we offer a conjecture and some remarks on the general case.

Notation.

(1) All functions to be considered are real; a, b denote real constants.

(2) (M, ρ) is a compact metric space.

(3) $\|f\| = \sup_{x \in M} |f(x)|$.

(4) If $g_1(x)$ and $g_2(x)$ are any functions on M,

$$E(g_1, g_2) = \sup_{f \in \mathscr{S}} \inf_{a,b} \| f - ag_1 - bg_2 \|.$$

(5) If $T \subseteq M$, T' is its complement, $d(T)$ is its diameter and

$$T(x) = \begin{cases} 1, & x \in T \\ 0, & x \in T'. \end{cases}$$

Now, we state our:

MAIN THEOREM. *There exists a $T \subseteq M$ such that*

$$E(T, T') = \inf_{g_1, g_2} E(g_1, g_2).$$

Moreover, we shall actually be able to calculate $E(T, T')$ in terms of the geometry of M.

THEOREM 1. *Let $g_1(x)$ and $g_2(x)$ be arbitrary functions on M. Then*

(1) *If $1 \in sp\{g_1, g_2\}$, $E(g_1, g_2) \leqslant 1/2\, d(M)$.*

(2) *If $1 \notin sp\{g_1, g_2\}$, $E(g_1, g_2) = \infty$.*

Proof. (1) For each $f \in \mathscr{S}$,

$$\| f(x) - 1/2[\sup f + \inf f] \| \leqslant 1/2\, d(M).$$

(2) If $1 \notin sp\{g_1, g_2\}$, let

$$\delta = \inf_{a,b} \| 1 - ag_1 - bg_2 \|.$$

By compactness (see [1], Lemma on p. 16), $\delta > 0$. Clearly,

$$\inf_{a,b} \| n - ag_1 - bg_2 \| = n\delta.$$

Hence, $E(g_1, g_2) \geqslant n\delta$ for all n, and therefore, $E(g_1, g_2) = \infty$. Q.E.D.

In looking for a best approximating space, we can assume 1 is in our space. We are now looking for a $g(x)$ such that $E(1, g)$ is a minimum.

Now that we have the constant function to approximate with, we can look only at those $f \in \mathcal{S}$ such that $\|f\| \leqslant d(M)$, since instead of f, we can deal with $f(x) - f(x_0)$, where $x_0 \in M$.

DEFINITION. $\mathcal{S}_0 = \{f \in \mathcal{S} : \|f\| \leqslant d(M)\}$.

In approximating a bounded function, we can always assume that our second function, g, is bounded. Without loss of generality, we can assume $\|g\| \leqslant 1$.

The next theorem provides a crucial inequality.

THEOREM 2. *Let x_1, x_2 and x_3 be points of M such that*

$$g(x_1) < g(x_2) < g(x_3).$$

Then

$$E(1, g) \geqslant \frac{[g(x_3) - g(x_2)]\,\rho(x_1, x_2) + [g(x_2) - g(x_1)]\,\rho(x_2, x_3)}{2[g(x_3) - g(x_1)]}.$$

Proof. Let $M_1 = \{x_1, x_2, x_3\}$ and let $\mathcal{S}(M_1)$ be the set of real valued functions $f(x)$ defined on M_1, such that

$$|f(x_i) - f(x_j)| \leqslant \rho(x_i, x_j), \qquad i, j = 1, 2, 3.$$

Let $f_0(x) \equiv \rho(x, x_2)$. It is easily seen that $f_0 \in \mathcal{S}_0 \subseteq \mathcal{S}(M_1)$. Let $a + bg(x)$ be a best approximation to f on M_1, and let $\delta = \|a + bg - f\|$, with the norm restricted to M_1. Then we have

$$a + bg(x_1) - f_0(x_1) = -\delta$$

$$a + bg(x_2) - f_0(x_2) = \delta$$

$$a + bg(x_3) - f_0(x_3) = -\delta.$$

(Essentially, the reason is that otherwise better a, b could be found.) Solving these three equations for δ, we have

$$\delta = \frac{[g(x_3) - g(x_2)]\,\rho(x_1, x_2) + [g(x_2) - g(x_1)]\,\rho(x_2, x_3)}{2[g(x_3) - g(x_1)]}.$$

Thus,

$$\inf_{a,b} \sup_{x \in M} |a + bg(x) - f_0(x)| \geqslant \inf_{a,b} \sup_{x \in M_1} |a + bg(x) - f_0(x)| = \delta$$

and, hence,

$$E(1, g) \geqslant \delta. \qquad \text{Q.E.D.}$$

The next theorem gives an explicit formula for $E(1, g)$, when g is a characteristic function.

THEOREM 3. *Let $T \subseteq M$. Then $E(1, T) = 1/2 \max[d(T), d(T')]$.*

Proof. Since $T'(x) = 1 - T(x)$, $sp\{1, T(x)\} = sp\{T(x), T'(x)\}$. Let $f \in \mathscr{S}_0$. We want to approximate f by $aT(x) + bT'(x)$, which is a if $x \in T$ and b if $x \in T'$. Let

$$a_0 = \tfrac{1}{2}[\sup_{x \in T} f(x) + \inf_{x \in T} f(x)],$$

$$b_0 = \tfrac{1}{2}[\sup_{x \in T'} f(x) + \inf_{x \in T'} f(x)].$$

For $t \in T$, we have

$$| f(t) - a_0 | = | \tfrac{1}{2}[f(t) - \sup_{x \in T} f(x)] + \tfrac{1}{2}[f(t) - \inf_{x \in T} f(x)] |.$$

The two summands never have the same sign and since $f \in \mathscr{S}_0$, each is less than $\frac{1}{2}d(T)$. Similarly, $| f(x) - b_0 | \leqslant \frac{1}{2}d(T')$ for $x \in T'$. Since f is arbitrary, $E(1, T) \leqslant \frac{1}{2} \max[d(T), d(T')]$.

Now let $\epsilon > 0$ be given, and choose $x_1, x_2 \in T$ such that

$$\rho(x_1, x_2) > d(T) - \epsilon.$$

Let $f_0(x) = \rho(x, x_1)$. Then $f_0 \in \mathscr{S}_0$. For any a, b,

$$\begin{aligned} \| f_0 - aT - bT' \| &\geqslant \max_{x = x_1, x_2} | f_0(x) - aT(x) - bT'(x) | \\ &= \max [| a |, | \rho(x_1, x_2) - a |] \\ &\geqslant 1/2\, \rho(x_1, x_2) \\ &\geqslant 1/2\, d(T) - \epsilon/2. \end{aligned}$$

Thus $E(1, T) > 1/2\, d(T) - \epsilon/2$. Similarly, $E(1, T) > 1/2\, d(T') - \epsilon/2$. Since ϵ was arbitrary, we are done.

THEOREM 4. *Let $g(x)$ have only a finite number of values. Then, there exists a $T \subseteq M$ such that $E(1, T) \leqslant E(1, g)$.*

Proof. Let $g(x)$ take on the values $y_1 \cdots y_n$, with $y_1 < y_2 < \cdots < y_n$. By the same argument used in the proof of the previous theorem,

$$d[g^{-1}(y_k)] \leqslant 2E(1, g), \qquad k = 1 \cdots n.$$

Let $U_1 = g^{-1}(y_1)$ and let U_k $(k > 1)$ be defined inductively as

$$U_k = U_{k-1} \cup \{x \in g^{-1}(y_k) : \rho(x, x') \leqslant 2E(1, g) \quad \text{for all} \quad x' \in U_{k-1}\}.$$

Obviously, $d(U_k) \leqslant 2E(1, g)$. Let $T = U_n$. Then $d(T) \leqslant 2E(1, g)$.

CLAIM. $d(T') \leqslant 2E(1, g)$.

To prove the claim, we take x_1 and $x_2 \in T'$ and assume $\rho(x_1, x_2) > 2E(1, g)$. For some j, k, $x_1 \in g^{-1}(y_j)$ and $x_2 \in g^{-1}(y_k)$, and since, $d[g^{-1}(y_j)] \leqslant 2E(1, g)$, we must have $j \neq k$; we can assume $j < k$. Since $g^{-1}(y_1) \subseteq T$, $1 < j < k$. As $x_1 \in T'$, there is an $x_0 \in g^{-1}(y_i)$ with $i < j$ and $\rho(x_0, x_1) > 2E(1, g)$.

By Theorem 2, since $g(x_0) < g(x_1) < g(x_2)$, we have

$$\begin{aligned} E(1, g) &\geqslant \frac{[g(x_2) - g(x_1)]\,\rho(x_1, x_0) + [g(x_1) - g(x_0)]\,\rho(x_1, x_2)}{2[g(x_2) - g(x_0)]} \\ &> \frac{[g(x_2) - g(x_1)]\,2E(1, g) + [g(x_1) - g(x_0)]\,2E(1, g)}{2[g(x_2) - g(x_0)]} \\ &= E(1, g) \end{aligned}$$

and we have proven the claim. Therefore, $1/2 \max[d(T), d(T')] \leqslant E(1, g)$ and by Theorem 3, $E(1, T) \leqslant E(1, g)$. Q.E.D.

We now eliminate the condition that $g(x)$ has only a finite number of values. For this, we need the following

LEMMA. *If $g(x)$ is non-constant and $\| g_n - g \| \to 0$, then $E(1, g_n) \to E(1, g)$.*

Proof. In approximating $f \in \mathscr{S}_0$ by $a + bg(x)$, we can always assume $\| a + bg - f \| \leqslant \| f \|$ (since, otherwise, we can do better with $a = b = 0$).

For every a, b, let $L(a + bg) = b$. Then L is a linear functional defined on a finite dimensional space and is, therefore, bounded, i.e., there exists a $c(g)$ such that

$$| b | = | L(a + bg) | \leqslant c(g) \| a + bg \|.$$

Let a and b be arbitrary and let $f \in \mathscr{S}_0$. Then

$$\begin{aligned} | \| a + bg - f \| - \| a + bg_n - f \| | &\leqslant \|(a + bg - f) - (a + bg_n - f)\| \\ &= | b | \, \| g - g_n \| \\ &\leqslant c(g) \| a + bg \| \, \| g - g_n \| \\ &\leqslant c(g)(\| a + bg - f \| + \| f \|) \, \| g - g_n \| \\ &\leqslant 2c(g) \| f \| \, \| g - g_n \| \\ &\leqslant 2c(g)\, d(M) \| g - g_n \|. \end{aligned}$$

Thus,

$$\| a + bg - f \| \leqslant \| a + bg_n - f \| + 2c(g)\, d(M) \| g - g_n \|.$$

Therefore,

$$\sup_{f \in \mathscr{S}_0} \inf_{a,b} \| a + bg - f \| \leqslant \sup_{f \in \mathscr{S}_0} \inf_{a,b} \| a + bg_n - f \| + 2c(g)\, d(M) \| g - g_n \|,$$

i.e., $E(1, g) \leqslant E(1, g_n) + 2c(g)\, d(M) \| g - g_n \|$. Similarly, starting with $\| a + bg_n - f \| \leqslant \| a + bg - f \| + 2c(g)\, d(M) \| g - g_n \|$, we get $E(1, g_n) \leqslant E(1, g) + 2c(g)\, d(M) \| g - g_n \|$. Letting $n \to \infty$, we have $E(1, g_n) \to E(1, g)$.
Q.E.D.

Assume that $g(x)$ has an infinite number of values.

Define $g_n(x)$ as follows: If $k/n < g(x) \leqslant k + 1/n$, let $g_n(x) = k + 1/n$. Since we assume that $\| g \| \leqslant 1$, $g_n(x)$ has at most $2n + 1$ values, $\| g_n(x) \| \leqslant 1$ and $\| g_n - g \| \leqslant 1/n$. For each n, there is a $T_n \subseteq M$ such that $E(1, T_n) \leqslant E(1, g_n)$. Therefore,

$$\underline{\lim}\, E(1, T_n) \leqslant \underline{\lim}\, E(1, g_n) = E(1, g).$$

Since,

$$\inf_n E(1, T_n) \leqslant \underline{\lim}\, E(1, T_n),$$

we have

$$\inf_n E(1, T_n) \leqslant E(1, g).$$

In the case where $g(x)$ has only a finite number of values, we rely on Theorem 4 to obtain a $T \subseteq M$ such that $E(1, T) \leqslant E(1, g)$.

Now, let G_1 be the set of all subsets of M and let G be the set of all functions on M. Then

$$\inf_{T \in G_1} E(1, T) \leqslant \inf_{g \in G} E(1, g)$$

and, since $G_1 \subseteq G$, we have

$$\inf_{T \in G_1} E(1, T) = \inf_{g \in G} E(1, g),$$

i.e., in choosing a best space we only have to look at characteristic functions! However, does there exist a $T_1 \in G_1$ such that

$$E(1, T_1) = \inf_{T \in G_1} E(1, T)\,?$$

In other words, does there exist a $T_1 \subseteq M$ such that

$$\max[d(T_1), d(T_1')] = \inf_{T \in G_1} \max[d(T), d(T')]\,?$$

This question is answered by the next theorem.

THEOREM 5. *Let $\{T_n\}$ be a sequence of subsets of M such that*

$$\lim_{n\to\infty} \{\max[d(T_n), d(T_n')]\} = 1.$$

Then there is a $T \subseteq M$ such that $\max[d(T), d(T')] \leqslant 1$.

Proof. Pick any $x_0 \in M$. By possibly interchanging T_n and T_n', we can assume $x_0 \in T_n$. Let

$$f_n(x) = \rho(x, T_n) = \inf_{y \in T_n} \rho(x, y).$$

Then $0 \leqslant f_n(x) \leqslant \rho(x, x_0) \leqslant d(M)$. Thus, $\{f_n\}$ is uniformly bounded. Now let $x, y \in M$ and $\delta > 0$ be given. There is a $y_1 \in T_n$ such that $|f_n(y) - \rho(y, y_1)| < \delta$. Then

$$\begin{aligned} f_n(x) - f_n(y) &\equiv \inf_{z \in T_n} \rho(x, z) - \inf_{z \in T_n} \rho(y, z) \\ &\leqslant \rho(x, y_1) - \rho(y, y_1) + \delta \\ &\leqslant \rho(x, y) + \delta. \end{aligned}$$

Similarly, we get $f_n(y) - f_n(x) \leqslant \rho(x, y) + \delta$. Since δ is arbitrary, we have $|f_n(x) - f_n(y)| \leqslant \rho(x, y)$ and, thus, $\{f_n\}$ is equicontinuous. By the Ascoli–Arzela Theorem, $\{f_n\}$ has a uniformly convergent subsequence which, without loss of generality, we can assume is $\{f_n\}$ itself, i.e., $f_n \to f$ uniformly. Let $T = f^{-1}(0)$. T is nonempty, since $x_0 \in T$.

CLAIM. $d(T) \leqslant 1$, $d(T') \leqslant 1$.

Proof. Take $x, y \in T$ so that $f(x) = 0$, $f(y) = 0$. Given $\delta > 0$, there is as N such that $n \geqslant N \Rightarrow f_n(x) < \delta$, $f_n(y) < \delta$ and $d(T_n) < 1 + \delta$. We can find $x_1, y_1 \in T_n$ such that $\rho(x, x_1) < \delta$, $\rho(y, y_1) < \delta$, implying

$$\rho(x, y) \leqslant \rho(x, x_1) + \rho(x_1, y_1) + \rho(y_1, y) < 1 + 3\delta.$$

Therefore, $d(T) \leqslant 1$.

Now take $x, y \in T'$ so that $f(x) \neq 0$, $f(y) \neq 0$. Given $\delta > 0$, there is an n such that $f_n(x) > 0$, $f_n(y) > 0$ and $d(T_n') < 1 + \delta$. Therefore, $\rho(x, T_n) > 0$ and $\rho(y, T_n) > 0$, i.e., $x, y \in T_n'$. Also $\rho(x, y) \leqslant d(T_n') < 1 + \delta$. Therefore, $d(T') \leqslant 1$. Q.E.D.

With Theorem 5 we have completed our proof of the Main Theorem. By Theorem 3, we know that the minimum of $E(g_1, g_2)$ is

$$1/2 \min_{T \subseteq M} [\max(d(T), d(T')].$$

If we are looking for a best one-dimensional approximating space, then as in Theorem 1, we see that the only such space is that of the constant function. The "error," $E(1, 0)$, is, by Theorem 3, $1/2\, d(M)$. What is surprising is that for many compact metric spaces M, the "error" in the two-dimensional case is the same as in the one-dimensional case.

Examples

(a) Let M be a closed equilateral triangle of side 1, in R^2. Then

$$E(1, 0) = (1/2)\, d(M) = 1/2.$$

Claim. *Let* $T \subseteq M$. *Then* $\max[d(T), d(T')] = 1$.

To prove the claim, we consider the three vertices. Either T or T' must contain at least two of the vertices; hence $\max[d(T), d(T')] = 1$. Therefore,

$$\inf_{g_1, g_2} E(g_1, g_2) = 1/2 \inf_T \{\max[d(T), d(T')]\} = 1/2 = E(1, 0).$$

(b) Let M be a closed regular pentagon in R^2 of side 1. Then

$$E(1, 0) = 1/2\, d(M) = \sqrt{1/2[1 - \cos(3\pi/5)]}.$$

Claim. *Let* $T \subseteq M$. *Then* $\max[d(T), d(T')] = d(M)$.

To prove the claim, assume it's false and consider the five vertices which (ordered in a clockwise manner) we denote by x_1, x_2, x_3, x_4, and x_5. Suppose $x_1 \in T$. Then, since $\rho(x_1, x_3) = \rho(x_1, x_4) = d(M)$, we must have x_3 and x_4 in T'. By the same argument, x_2 and x_5 must be in T. But $\rho(x_2, x_5) = d(M)$, proving our claim. Therefore,

$$\inf_{g_1, g_2} E(g_1, g_2) = (1/2) \inf_T \{\max[d(T), d(T')]\} = 1/2\, d(M) = E(1, 0).$$

This example generalizes to any closed regular polygon in R^2 of sidelength one, with an *odd* number of sides.

(c) Let M be a closed disk in R^2 of diameter 1. Then

$$E(1, 0) = (1/2)\, d(M) = 1/2.$$

Claim. *Let* $T \subseteq M$. *Then* $\max[d(T), d(T')] = 1$.

To prove this claim, assume it's false and consider the points on the boundary of M. If such a point p belongs to T, then its antipodal point p', as well as some neighborhood N' of p', is in T'. If N is the set of antipodal points of the points of N', then N, as well as some neighborhood of N, is contained in T. Continuing in this manner, we end up dividing the circle into two disjoint, open sets, which is impossible. We have thus proven the claim.

Thus,

$$\inf_{g_1, g_2} E(g_1, g_2) = \inf_T E(1, T) = (1/2) \inf_T \{\max[d(T), d(T')]\} = 1/2 = E(1, 0).$$

This example generalizes to a closed ball in R^n ($n \geqslant 2$) of diameter one.

Remarks. If $M = [0, 1]$, then, as we saw, we can decompose M into n pairwise disjoint sets, $T_1, \ldots, T_n$, such that

$$E(T_1, \ldots, T_n) = \inf_{g_1, \ldots, g_n} E(g_1, \ldots, g_n). \qquad (*)$$

If M is an arbitrary compact metric space, then, as we have seen, we can find disjoint sets T_1 and $T_2 (= T_1')$ such that

$$E(T_1, T_2) = \inf_{g_1, g_2} E(g_1, g_2).$$

It has been conjectured that in this general case for every n, M can be decomposed into n pairwise disjoint sets $T_1, T_2, \ldots, T_n$ such that (*) holds. However, for $n = 3$ and M, a closed square in R^2, we have disproven this conjecture. We offer here a weaker conjecture: If M is a compact metric space, and $n \geqslant 1$, there are subsets $T_1, \ldots, T_n$ (not necessarily pairwise disjoint) such that (*) holds. If true, it would establish that there always exists a best approximating space which is spanned by characteristic functions, but it would not be as easy to calculate the "error" as in Theorem 3.

References

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